APPROXIMATE SOLUTION OF THE PROBLEM OF THE THREE-DIMENSIONAL BOUNDARY LAYER IN AN INCOMPRESSIBLE FLUID

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A method of successive approximations is proposed for the solution of the equations of the three-dimensional incompressible boundary layer on bodies of arbitrary shape. A coordinate system connected with the streamlines of the external nonviscous flow is used. It is assumed that the velocity across the external streamlines is small. When the intensity of secondary flow is low the equations describing the boundary layer in an incompressible fluid are reduced to a form analogous to the equations for the boundary layer on axially symmetrical bodies. An approximate analytical solution is obtained for the velocity and for the friction in the form of equations which can be used for any problems of a three-dimensional incompressible boundary layer. The method developed was applied to the problem of the three-dimensional boundary layer at a plate with a cylindrical obstacle in the presence of a slip angle.

1. Let us consider the three-dimensional laminar boundary layer in an incompressible fluid at an arbitrary surface. We will use the curvilinear orthogonal coordinate system ξ, η, ζ, connected with the streamlines of the external ideal flow at the surface. The coordinate ζ is the distance from the surface of the body along the normal, so that $\zeta=0$ is the equation of the surface in the flow, the lines η =const are the streamlines of the nonviscous flow at the surface, and the lines ξ =const are their orthogonal trajectories, i.e., the equipotential lines.

The equations of the three-dimensional incompressible boundary layer in the coordinate system chosen chosen have the form [1]

$$\frac{u}{\sqrt{g_{11}}}\frac{\partial u}{\partial \xi}+\frac{w}{\sqrt{g_{22}}}\frac{\partial u}{\partial \eta}+v\frac{\partial u}{\partial \zeta}-\frac{1}{\sqrt{g_{11}g_{22}}}\frac{\partial \sqrt{g_{22}}}{\partial \xi}w^2+\frac{1}{\sqrt{g_{11}g_{22}}}\frac{\partial \sqrt{g_{11}}}{\partial \eta}uw=\frac{U_e}{\sqrt{g_{11}}}\frac{\partial U_e}{\partial \xi}+\nu\frac{\partial^2 u}{\partial \zeta^2}$$

$$\frac{u}{\sqrt{g_{11}}}\frac{\partial w}{\partial \xi}+\frac{w}{\sqrt{g_{22}}}\frac{\partial w}{\partial \eta}+v\frac{\partial w}{\partial \zeta}-\frac{1}{\sqrt{g_{11}g_{22}}}\frac{\partial \sqrt{g_{11}}}{\partial \eta}u^2+\frac{1}{\sqrt{g_{11}g_{22}}}\frac{\partial \sqrt{g_{22}}}{\partial \xi}uw=-\frac{1}{\sqrt{g_{11}g_{22}}}\frac{\partial \sqrt{g_{11}}}{\partial \eta}U_e^2+\nu\frac{\partial^2 w}{\partial \zeta^2}, \quad (1)$$

$$\frac{\partial}{\partial \xi}(\sqrt{g_{22}}u)+\frac{\partial}{\partial \eta}(\sqrt{g_{11}}w)+\sqrt{g_{11}g_{22}}\frac{\partial v}{\partial \zeta}=0$$

Here u, w, and v are the projections of the velocity on the coordinate lines ξ, η, and ζ, respectively, ν is the kinematic viscosity coefficient, and $g_{11}(\xi, \eta)$ and $g_{22}(\xi, \eta)$ are metric coefficients.

The boundary conditions for system (1.1) are chosen as follows:

$$u=w=v=0 \text{ at } \zeta=0,\ u\to U_e(\xi,\eta),\ w\to 0 \text{ as } \zeta\to\infty \quad (1.2)$$

Here U_e is the total velocity of nonviscous flow at the surface of the body.

As the coordinate of ξ along the streamlines we take the potential φ of the external ideal flow. We denote the coordinate of η orthogonal to the streamlines through ψ. If ξ_1 and η_1 are an arbitrary orthogonal coordinate system at the surface then for the components of the velocity of the external flow we have

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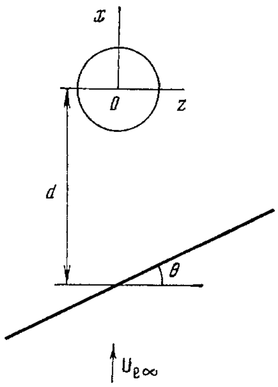

Fig. 1

$$U_e = \frac{1}{\sqrt{g_{11}}}\frac{\partial\varphi}{\partial\xi_1}, \qquad w_e = \frac{1}{\sqrt{g_{22}}}\frac{\partial\varphi}{\partial\eta_1}$$

If we take φ and ψ as ξ_1 and η_1 we obtain

$$U_e = 1/\sqrt{g_{11}}$$

The element of length at the surface can be represented in the form

$$ds^2 = \frac{1}{U_e^2}\left(d\varphi^2 + \frac{1}{r^2}d\psi^2\right), \qquad g_{11} = 1/U_e^2, \ g_{22} = 1/(r^2U_e^2) \tag{1.3}$$

where $r^2 = 1/(g_{22}U_e^2)$ is a function which depends on the geometry of the surface, the external flow, and the choice of the coordinate ψ.

We will assume that the velocity across the streamlines of the external flow and its derivatives are small compared with the velocity along the streamlines. Since the transverse velocity is equal to zero at the surface of the body and at the outer boundary of the boundary layer, the assumption that the secondary flow is of low intensity is justified to a certain extent [2, 3]. The secondary flow is small when the external streamlines have small geodesic curvature. Up to values of 0.3-0.4 for the ratio of the velocity of transverse flow in the boundary layer to the total local velocity one can use the principle of the independence of the longitudinal flow from the transverse flow.

In (1.1) let us make the substitution

$$\zeta_1 = \sqrt{U_e}\,\zeta, \quad v_1 = v/\sqrt{U_e}, \quad r_1 = 1/rU \tag{1.4}$$

On the assumption that the secondary flow is small we arrive at equations analogous to the equations for the boundary layer near an axially symmetrical body from which is found the longitudinal velocity and a linear equation for the transverse velocity

$$u\frac{\partial u}{\partial\xi} + v_1\frac{\partial u}{\partial\zeta_1} = U_e\frac{\partial U_e}{\partial\xi} + \nu\frac{\partial^2 u}{\partial\zeta_1^2}, \quad \frac{\partial}{\partial\xi}(r_1 u) + r_1\frac{\partial v_1}{\partial\zeta_1} = 0 \tag{1.5}$$

$$u\frac{\partial w}{\partial\xi} + v_1\frac{\partial w}{\partial\zeta_1} + \frac{\partial\ln r_1}{\partial\xi}uw + \frac{1}{r_1U_e}\frac{\partial\ln U_e}{\partial\eta}(u^2 - U_e^2) = \nu\frac{\partial_2 w}{\partial\zeta_1^2} \tag{1.6}$$

with the boundary conditions

$$u = w = v_1 = 0 \ \text{ at } \ \zeta_1 = 0, \ u \to U_e, \ w \to 0 \ \text{ as } \ \zeta_1 \to \infty$$

2. Let us find the function r. Let $z = f(x, y)$ be the equation in Cartesian coordinates for the surface over which the flow occurs. Let us change from the coordinates x, y, z to the coordinates φ, ψ, ζ. For this we construct the normal from the point (x, y, z) to the surface of the body. We designate the Cartesian coordinates of the point of intersection as x_0, y_0, $z_0 = f(x_0, y_0)$.

Then φ, ψ, ζ are connected with x, y, z by the equations

$$\varphi = \varphi(x_0, y_0, f(x_0, y_0)), \ \psi = \psi(x_0, y_0, f(x_0, y_0))$$
$$\zeta = |z - f(x_0y_0)|\sqrt{1 + (\partial f/\partial x_0)^2 + (\partial f/\partial y_0)^2}$$

where x_0 and y_0 are found from the equations

$$x_0 = x + (z - f(x_0, y_0))\,\partial f/\partial x_0, \ y_0 = y + (z - f(x_0, y_0))\,\partial f/\partial y_0$$

It is necessary to find g_{22} at $z = f(x, y)$

$$g_{22} = \left(\frac{\partial x}{\partial\psi}\right)^2 + \left(\frac{\partial y}{\partial\psi}\right)^2 + \left(\frac{\partial z}{\partial\psi}\right)^2 = \frac{1}{g^{22}} = \left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right]^{-1}$$

Since the velocity vector of the external ideal flow $\mathbf{U_e}$ $[u_x, u_y, u_z]$ lies in a plane tangent to the surface in the flow,

$$u_z = \frac{\partial f}{\partial x_0}u_x + \frac{\partial f}{\partial y_0}u_y$$

Since the lines ψ =const, $z = f(x, y)$ are streamlines at the surface of the body it is easy to find that

$$\frac{\partial\psi}{\partial y_0} = -\frac{u_x}{u_y}\frac{\partial\psi}{\partial x_0} \tag{2.1}$$

With this in mind one can find

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial x_0}\left(\frac{\partial x_0}{\partial x} - \frac{u_x}{u_y}\frac{\partial y_0}{\partial x}\right) = -\frac{\partial\psi}{\partial x_0}\frac{1}{u_y}\left[1+\left(\frac{\partial f}{\partial x_0}\right)^2 + \left(\frac{\partial f}{\partial y_0}\right)^2\right]^{-1}\left[-u_y\left(1+\left(\frac{\partial f}{\partial y_0}\right)^2\right) - u_x\frac{\partial f}{\partial x_0}\frac{\partial f}{\partial y_0}\right]$$

$$\frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial x_0}\left(\frac{\partial x_0}{\partial y} - \frac{u_x}{u_y}\frac{\partial y_0}{\partial y}\right) = -\frac{\partial\psi}{\partial x_0}\frac{1}{u_y}\left(1+\left(\frac{\partial f}{\partial x_0}\right)^2 + \left(\frac{\partial f}{\partial y_0}\right)^2\right)^{-1}\left[u_y\frac{\partial f}{\partial x_0}\frac{\partial f}{\partial y_0} + u_x\left(1+\left(\frac{\partial f}{\partial x_0}\right)^2\right)\right]$$

$$\frac{\partial\psi}{\partial z} = \frac{\partial\psi}{\partial x_0}\left(\frac{\partial x_0}{\partial z} - \frac{u_x}{u_y}\frac{\partial y_0}{\partial z}\right) = -\frac{\partial\psi}{\partial x_0}\frac{1}{u_y}\left(1+\left(\frac{\partial f}{\partial x_0}\right)^2 + \left(\frac{\partial f}{\partial y_0}\right)^2\right)^{-1}\left[-u_y\frac{\partial f}{\partial x_0} + u_x\frac{\partial f}{\partial y_0}\right]$$

Thus, we obtain

$$\begin{gathered} g_{22} = \frac{1}{U_e^2}\left(1+\left(\frac{\partial f}{\partial x_0}\right)^2 + \left(\frac{\partial f}{\partial y_0}\right)^2\right)\left(\frac{u_y}{\partial\psi/\partial x_0}\right)^2 \\ r = \left|\frac{\partial\psi/\partial x_0}{u_y}\left(1+\left(\frac{\partial f}{\partial x_0}\right)^2 + \left(\frac{\partial f}{\partial y_0}\right)^2\right)^{-1/2}\right| \end{gathered} \tag{2.2}$$

If $\mathbf{U}_e$ (x, y, z) is known one can find $\partial U_e/\partial\xi$ and $\partial U_e/\partial\eta$ ($\xi = \varphi$, $\eta = \psi$) from the equations

$$\begin{gathered} U_e^2\frac{\partial U_e}{\partial\xi} = u_x\frac{\partial U_e}{\partial x} + u_y\frac{\partial U_e}{\partial y} + u_z\frac{\partial U_e}{\partial z} \\ -U_e^2\frac{\partial\psi/\partial x_0}{u_y}\frac{\partial U_e}{\partial\eta} = -\left(u_y + \frac{\partial f}{\partial y}u_z\right)\frac{\partial U_e}{\partial x} + \left(u_x + \frac{\partial f}{\partial x}u_z\right)\frac{\partial U_e}{\partial y} + \left(u_x\frac{\partial f}{\partial y} - u_y\frac{\partial f}{\partial x}\right)\frac{\partial U_e}{\partial z} \end{gathered} \tag{2.3}$$

3. Let us introduce new variables by analogy with [1]

$$\begin{gathered} \lambda = \left[\frac{U_e(\xi,\eta)}{\nu[\xi - \xi_0(\eta)]}\right]^{1/2}\zeta_1 \\ u = U_e(\xi,\eta)E(\xi,\eta,\lambda), \quad w = U_e(\xi,\eta)G(\xi,\eta,\lambda) \\ v_1 = \left[\frac{\nu U_e(\xi,\eta)}{\xi - \xi_0(\eta)}\right]^{1/2}\left[K(\xi,\eta,\lambda) - [\xi - \xi_0(\eta)]E(\xi,\eta,\lambda)\frac{\partial\lambda}{\partial\xi} - r(\xi,\eta)[\xi - \xi_0(\eta)]G(\xi,\eta,\lambda)\frac{\partial\lambda}{\partial\eta}\right] \end{gathered} \tag{3.1}$$

where $\xi_0(\eta) = \varphi_0(\psi)$ is the equation of the boundary of the body from which the boundary layer begins to develop. For example, in the case of flow over a flat plate with an obstacle mounted in front of it $\xi_0(\eta)$ is the equation of the leading edge of the plate. If one is considering the flow over a blunt body where there is a critical point, then the function $\xi_0(\eta) = 0$. The potential φ is reckoned from this leading critical point, i.e., at it $\varphi = \xi = 0$.

Equations (1.5) and (1.6) for u, v_1, and w are transformed into equations for E, K, and G

$$\begin{gathered} \frac{\partial^2 E}{\partial\lambda^2} = K\frac{\partial E}{\partial\lambda} + N_1(E^2 - 1) + N_4 E\frac{\partial E}{\partial\xi}, \quad \frac{\partial K}{\partial\lambda} = -P_1 E - N_4\frac{\partial E}{\partial\xi} \\ \frac{\partial^2 G}{\partial\lambda^2} = K\frac{\partial G}{\partial\lambda} + M_1(E^2 - 1) + M_3 EG + N_4 E\frac{\partial G}{\partial\xi} \end{gathered} \tag{3.2}$$

with the boundary conditions

$$E = G = K = 0 \quad \text{at } \lambda = 0,\ E \to 1,\ G \to 0 \text{ as } \lambda \to \infty \tag{3.3}$$

The coefficients P_1, M_1, M_3, N_1, and N_4 are known functions of ξ and η and are determined by the geometry of the surface and the external flow

$$\begin{gathered} N_1 = \partial\ln U_e/\partial\ln\xi_1, \quad N_4 = \xi_1, \quad M_1 = \xi_1 r\,\partial\ln U_e/\partial\eta \\ M_3 = -\partial\ln r/\partial\ln\xi_1, \quad P_1 = \partial\ln\frac{\sqrt{\xi_1}}{rU_e}\bigg/\partial\ln\xi_1, \quad \xi_1 = \xi - \xi_0(\eta) \end{gathered} \tag{3.4}$$

The components of the friction at the surface of the body in the flow are found from the equations

$$\tau_1 = \mu \frac{\partial u}{\partial \zeta} = \mu U_e \frac{\partial E}{\partial \lambda} \frac{\partial \lambda}{\partial \zeta} = \sqrt{\frac{\mu\rho}{\xi - \xi_0}} U_e^2 \frac{\partial E}{\partial \lambda}$$
$$\tau_2 = \mu \frac{\partial w}{\partial \zeta} = \mu U_e \frac{\partial G}{\partial \lambda} \frac{\partial \lambda}{\partial \zeta} = \sqrt{\frac{\mu\rho}{\xi - \xi_0}} U_e^2 \frac{\partial G}{\partial \lambda} \qquad (3.5)$$

Here μ is the dynamic viscosity coefficient and ρ is the density.

4. Let us obtain integral equations for the functions E and G by integrating the corresponding differential equations twice with respect to λ (from λ to ∞ and then from 0 to λ)

$$-E = P_1\theta_{01}^* + (P_1 + N_1)\theta_{11}^* + N_1\theta_1^* + N_4 \frac{\partial \theta_{11}^*}{\partial \xi} + N_4\theta_{03}^*$$
$$-G = P_1\theta_{02}^* + M_1\theta_{11}^* + (P_1 + M_3)\theta_{21}^* + M_1\theta_1^* + N_4 \frac{\partial \theta_{21}^*}{\partial \xi} + N_4\theta_{04}^* \qquad (4.1)$$

For the dimensionless friction at the wall we have the following equations

$$-\frac{\partial E}{\partial \lambda}\Big|_{\lambda=0} = (P_1 + N_1)\theta_{11} + N_1\theta_1 + N_4 \frac{\partial \theta_{11}}{\partial \xi}$$
$$-\frac{\partial G}{\partial \lambda}\Big|_{\lambda=0} = M_1\theta_{11} + (P_1 + M_3)\theta_{21} + M_1\theta_1 + N_4 \frac{\partial \theta_{21}}{\partial \xi} \qquad (4.2)$$

$$\theta_{11} = \int_\lambda^\infty (E-1)E\,d\lambda, \quad \theta_{21} = \int_\lambda^\infty EG\,d\lambda, \quad \theta_1 = \int_\lambda^\infty (E-1)\,d\lambda \qquad (4.3)$$

$$\theta_{11}^* = \int_0^\lambda\int_\lambda^\infty (E-1)E\,d\lambda\,d\lambda, \quad \theta_{21}^* = \int_0^\lambda\int_\lambda^\infty EG\,d\lambda\,d\lambda, \quad \theta_1^* = \int_0^\lambda\int_\lambda^\infty (E-1)\,d\lambda\,d\lambda$$

$$\theta_{01}^* = \int_0^\lambda \Big[(E-1)\int_0^\lambda E\,d\lambda\Big] d\lambda, \quad \theta_{02}^* = \int_0^\lambda \Big[G\int_0^\lambda E\,d\lambda\Big] d\lambda$$

$$\theta_{03}^* = \int_0^\lambda \Big[(E-1)\frac{\partial}{\partial \xi}\int_0^\lambda E\,d\lambda\Big] d\lambda, \quad \theta_{04}^* = \int_0^\lambda \Big[G\frac{\partial}{\partial \xi}\int_0^\lambda E\,d\lambda\Big] d\lambda$$

We will solve the integral equations (4.1) by the method of successive approximations [4]. For this it is necessary that after the substitution into the right sides of these equations of the arbitrary integrable functions E (ξ, η, λ) and G (ξ, η, λ) which satisfy the boundary conditions

$$E(\xi, \eta, 0) = G(\xi, \eta, 0) = 0, \; E(\xi, \eta, \infty) = 1, \; G(\xi, \eta, \infty) = 0$$

the right sides must take on values equal to unity and zero, respectively, as $\lambda \to \infty$. For this we introduce the unknown "correcting" functions δ (ξ, η) and b (ξ, η)

$$E^{(n)} = E^{(n)}(\xi, \eta, \lambda / \sqrt{\delta^{(n+1)}}), \quad G^{(n)} = b^{(n+1)}G^{(n)}(\xi, \eta, \lambda / \sqrt{\delta^{(n+1)}}) \qquad (4.4)$$

and the new independent variable

$$\zeta^{(n)} = \lambda / \sqrt{\delta^{(n+1)}} \qquad (4.5)$$

After the substitution into the right sides of (4.1) of the function $E^{(n)}$ $[\xi, \eta, \zeta^{(n)}]$ in place of E and the function $b^{(n+1)}$ $G^{(n)}$ $[\xi, \eta, \zeta^{(n)}]$ in place of G and the change from integration over λ to integration over $\zeta^{(n)}$ we obtain the connection between the (n+1)-th approximation and the n-th approximation

$$-E^{(n+1)} = \delta^{(n+1)}A^{(n)} + a_1^{(n)} \frac{d\delta^{(n+1)}}{d\xi}$$
$$-G^{(n+1)} = \delta^{(n+1)}(B^{(n)} + b^{(n+1)}C^{(n)}) + a_2^{(n)}b^{(n+1)} \frac{d\delta^{(n+1)}}{d\xi} + a_3^{(n)}\delta^{(n+1)} \frac{db^{(n+1)}}{d\xi} \qquad (4.6)$$

Here $A^{(n)}$ $[\xi, \eta, \zeta^{(n)}]$, $B^{(n)}$ $[\xi, \eta, \zeta^{(n)}]$, $C^{(n)}$ $[\xi, \eta, \zeta^{(n)}]$, $a_i^{(n)}$ $[\xi, \eta, \zeta^{(n)}]$ (i=1, 2, 3) are known functions of $E^{(n)}$ and $G^{(n)}$.

We choose the functions $\delta^{(n+1)}$ (ξ, η) and $b^{(n+1)}$ (ξ, η) in such a way that the boundary conditions are satisfied at the outer boundary of the boundary layer for $E^{(n+1)}$ and $G^{(n+1)}$. As $\zeta \to \infty$ we have $E^{(n+1)} \to 1$ and $G^{(n+1)} \to 0$. Hence we obtain linear first-order differential equations for the determination of $\delta^{(n+1)}$ and $b^{(n+1)}$

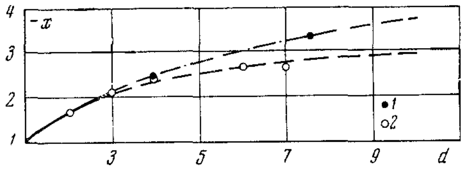

Fig. 2

$$a_{1\infty}^{(n)}\frac{d\delta^{(n+1)}}{d\xi} = -A_{\infty}^{(n)}\delta^{(n+1)} - 1$$
$$a_{3\infty}^{(n)}\frac{db^{(n+1)}}{d\xi} = b^{(n+1)}\left(-C_{\infty}^{(n)} + A_{\infty}^{(n)}\frac{a_{2\infty}^{(n)}}{a_{1\infty}^{(n)}} + \frac{a_{2\infty}^{(n)}}{a_{1\infty}^{(n)}}\frac{1}{\delta^{(n+1)}}\right) - B_{\infty}^{(n)} \tag{4.7}$$

These equations can be integrated. First the first equation is solved, the $\delta^{(n+1)}$ obtained is substituted into the second equation, and then $b^{(n+1)}$ is determined.

The (n + 1)-th approximation for the dimensionless friction at the wall is found from the equations

$$-\frac{\partial E}{\partial\lambda}\Big|_{\lambda=0}^{(n+1)} = \sqrt{\delta^{(n+1)}}\left[N_1(\theta_{11}^{(n)} + \theta_1^{(n)}) + P_1\theta_{11}^{(n)} + N_4\frac{\partial\theta_{11}^{(n)}}{\partial\xi}\right] + N_4\theta_{11}^{(n)}\frac{d\sqrt{\delta^{(n+1)}}}{d\xi}$$
$$-\frac{\partial G}{\partial\lambda}\Big|_{\lambda=0}^{(n+1)} = \sqrt{\delta^{(n+1)}}\left[M_1(\theta_{11}^{(n)} + \theta_1^{(n)}) + b^{(n)}\left((P_1 + M_3)\theta_{21}^{(n)} + N_4\frac{\partial\theta_{21}^{(n)}}{\partial\xi}\right)\right] + N_4\theta_{21}^{(n)}\left(b^{(n+1)}\frac{d\sqrt{\delta^{(n+1)}}}{d\xi} + \sqrt{\delta^{(n+1)}}\frac{db^{(n+1)}}{d\xi}\right) \tag{4.8}$$

Given the zeroth approximation by some means one can determine all the subsequent approximations for the components of the velocity and friction at the surface using (4.6)-(4.8).

5. Let us examine the first approximation. We give the zeroth approximation in the form

$$E^{(0)} = 1 - Z_0(\zeta^{(0)}),\quad G^{(0)} = b^{(1)}(Z_0(\zeta^{(0)}) - Z_{-1}(\zeta^{(0)}))$$
$$Z_0(\zeta) = -\frac{2}{\sqrt{\pi}}\int_{\infty}^{\zeta} e^{-\xi^2}d\xi,\quad Z_{-1}(\zeta) = e^{-\zeta^2} \tag{5.1}$$

The subsequent approximations $E^{(n)}$ and $G^{(n)}$ will be connected with a class of functions Z_m [5]

$$Z_m(\zeta) = \frac{A_m}{m!}\int_{\infty}^{\zeta}(\zeta - \xi)^m e^{-\xi^2}d\xi,\quad m = 0, 1, 2, \ldots \tag{5.2}$$

Here the A_m are chosen so that $Z_m(0) = 1$.

The functions $\delta(\xi, \eta)$ and $b(\xi, \eta)$ in the first approximation are found from the equations

$$N_4\frac{d\delta^{(1)}}{d\xi_1} = -\delta^{(1)}\left[2P_1 + \left(\frac{4}{\pi} + 2\right)N_1\right] + 8 \tag{5.3}$$

$$N_4\frac{db^{(1)}}{d\xi_1} = b^{(1)}\left[-M_3 + 3(1 - \sqrt{2})\frac{2+\pi}{2 - \sqrt{2}\pi}N_1 + 3(\sqrt{2} - 1)\frac{4\pi}{2 - \sqrt{2}\pi}\frac{1}{\delta^{(1)}}\right] + \frac{2+\pi}{2 - \sqrt{2}\pi}M_1 \tag{5.4}$$

After substituting the values of the coefficients Eq. (5.3) takes the form

$$\frac{d\delta^{(1)}}{d\ln\xi_1} + \delta^{(1)}\frac{d}{d\ln\xi_1}\left(\ln\frac{\xi_1 U_e^{4/\pi}}{r^2}\right) = 8$$

The initial condition for this equation is found from the requirement that $d\delta^{(1)}/d\xi_1$ be finite when $\xi_1 = 0$. For this it is necessary that

$$\delta^{(1)}|_{\xi_1=0} = 8$$

Solving the equation for $\delta^{(1)}$ with this initial condition we obtain

$$\delta^{(1)} = \frac{8}{M}\int_0^{\xi_1}\frac{M}{\xi_1}d\xi_1 = \frac{8}{M}\int_{\xi_0}^{\xi}\frac{M}{\xi - \xi_0}d\xi,\quad M = \frac{\xi_1}{r^2}U_e^{4/\pi} = \frac{\xi - \xi_0}{r^2}U_e^{4/\pi} \tag{5.5}$$

Disclosing the values of the coefficients and using the expression found for $\delta^{(1)}$, we have an equation for the function $b^{(1)}(\xi, \eta)$

$$\frac{db^{(1)}}{d\ln\xi_1} = b^{(1)}\frac{d}{d\ln\xi_1}\ln\left[rU_e^{2.616}\left(\int_0^{\xi_1}\frac{M}{\xi_1}d\xi_1\right)^{-0.7992}\right] - 2.1048\xi_1 r\frac{\partial\ln U_e}{\partial\eta}$$

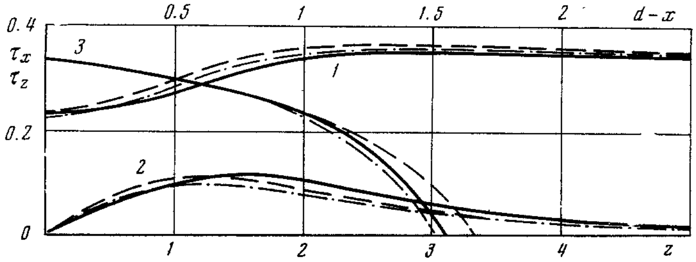

Fig. 3

From the assumption that $db^{(1)}/d\xi_1$ is finite at $\xi_1=0$ we obtain the initial condition for $b^{(1)}$: $b^{(1)}|_{\xi_1=0}=0$. The solution of the equation with this initial condition has the form

$$b^{(1)} = -\frac{2.1048r}{N}\int_0^{\xi_1}\frac{\partial \ln U_e}{\partial \eta} N\, d\xi_1, \quad N = \left(\int_0^{\xi_1}\frac{M}{\xi_1}d\xi_1\right)^{0.7992} U_e^{-2.616} \tag{5.6}$$

Knowing $\delta^{(1)}$ and $b^{(1)}$ one can find the first approximation for the velocity components and for the friction at the wall. With allowance for (4.6) and (5.3), (5.4) we obtain for $E^{(1)}$ and $G^{(1)}$ the expressions

$$E^{(1)} = \delta^{(1)}\frac{\partial \ln U_e}{\partial \ln \xi_1}\left[a_1\left(\frac{4}{\pi}+2\right) - J_{00} - \frac{1}{2}(Z_2-1)\right] - 8a_1$$

$$G^{(1)} = b^{(1)}\left[3(\sqrt{2}-1)\frac{4\pi}{\sqrt{2\pi}-2}a_3 - 8a_2\right] + \delta^{(1)} r\xi_1\frac{\partial \ln U_e}{\partial \eta}\left[\frac{2+\pi}{\sqrt{2\pi}-2}a_3 - J_{0.0} - \frac{1}{2}(Z_2-1)\right] + \tag{5.7}$$
$$+\delta^{(1)}b^{(1)}\frac{\partial \ln U_e}{\partial \ln \xi_1}\left[\left(\frac{4}{\pi}+2\right)a_2 - 3(\sqrt{2}-1)\frac{2+\pi}{\sqrt{2\pi}-2}a_3\right]$$

$$a_1 = \frac{a_1^{(0)}}{N_4} = \frac{1}{2}\left[J_{0.0} - \frac{1}{2\pi}(1-Z_1^2) + \frac{1}{\pi}(1-Z_1) - \frac{1}{2}(1-Z_3)\right]$$

$$a_2 = \frac{a_2^{(0)}}{N_4} = \frac{1}{2}\left[\frac{1}{2\pi}(1-Z_1^2) - I^*_{-1.1} + I^*_{0.0} - J_{0.0} - \frac{1}{2}(1-Z_{-1}) + (1-Z_0) - \left(\frac{1}{\pi}+\frac{1}{2}\right)(1-Z_1)\right] \tag{5.8}$$

$$a_3 = a_3^{(0)}/N_4 = {}^1/_4(1-Z_2) - {}^1/_2(1-Z_1) - J_{0.0} + I^*_{0.0}$$

$$J_{0.0} = \frac{1}{2\pi}(1-Z_1^2) + \frac{1}{4}(1-Z_0^2) - \frac{1}{\pi}(1-Z_1(\sqrt{2}\,\zeta^{(1)}))$$

$$I^*_{0.0} = \frac{1}{2} - \frac{1}{4}Z_0(Z_1+Z_{-1}) - \frac{1}{2\sqrt{2}}(1-Z_0(\sqrt{2}\,\zeta^{(1)}))$$

$$I^*_{-1.1} = \frac{1}{2}Z_0(Z_{-1}-Z_1) + \frac{1}{\sqrt{2}}(1-Z_0(\sqrt{2}\,\zeta^{(1)}))$$

Let us find the values of the dimensionless friction at the surface of the body in the first approximation. Using (4.8), (5.3), and (5.4) we have

$$\left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=0}^{(1)} = \sqrt{\delta^{(1)}}\frac{\partial \ln U_e}{\partial \ln \xi_1}\left(\frac{1}{\sqrt{\pi}} - \frac{2(\sqrt{2}-1)}{\pi\sqrt{\pi}}\right) + \frac{1}{\sqrt{\delta^{(1)}}}\frac{4(\sqrt{2}-1)}{\sqrt{\pi}}$$

$$\left.\frac{\partial G}{\partial \lambda}\right|_{\lambda=0}^{(1)} = \xi_1 r\frac{\partial \ln U_e}{\partial \eta}\sqrt{\delta^{(1)}}\left[\sqrt{\frac{2}{\pi}} + \frac{2+\pi}{2-\sqrt{2\pi}}\left(\frac{\sqrt{\pi}}{4} - \frac{\sqrt{2}-1}{\sqrt{\pi}}\right)\right] +$$
$$+ b^{(1)}\left(\frac{\sqrt{\pi}}{4} - \frac{\sqrt{2}-1}{\sqrt{\pi}}\right)\left[\sqrt{\delta^{(1)}}\frac{\partial \ln U_e}{\partial \ln \xi_1}(2+\pi)\left(\frac{3(\sqrt{2}-1)}{\sqrt{2}\pi-2} - \frac{1}{\pi}\right) + \frac{4}{\sqrt{\delta^{(1)}}}\left(1 - \frac{3(\sqrt{2}-1)\pi}{\sqrt{2}\pi-2}\right)\right] \tag{5.9}$$

6. Let us use the proposed method to examine the problem of the boundary layer formed at a thin semiinfinite plate perpendicular to which is placed an infinite cylinder, with an incompressible fluid flowing over the plate.

We will consider flow over the plate at a slip angle θ, where θ is the angle between the normal to the velocity of the impinging stream at infinity and the leading edge of the plate (Fig. 1).

In the case of a circular cylinder the relative potential of the external ideal flow is taken as the coordinate φ [6]

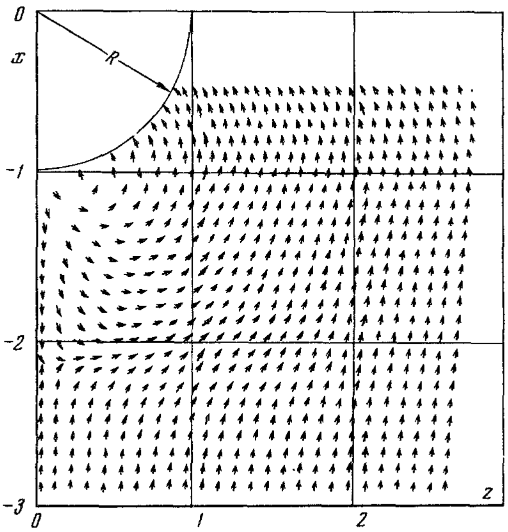

Fig. 4

$$\xi = \varphi = x\left(1 + \frac{1}{x^2 + z^2}\right) + \frac{d^2 + 1}{d} \tag{6.1}$$

Here x and z are the Cartesian coordinates in the plane, relative to the radius of the cylinder. Since the nonviscous flow is two-dimensional in the present problem, one can take as the coordinate ψ the flow function

$$\eta = \psi = z\left(1 - 1/(x^2 + z^2)\right) \tag{6.2}$$

Then

$$r \equiv 1, \quad g_{11} = g_{22} = 1 / U_e^2 \tag{6.3}$$

In the coordinates x and z the equation of the leading edge of the plate has the form $x + d = z \tan\theta$. Let us write this equation in the coordinates ψ, φ

$$\begin{gathered} \varphi_0 = (z \operatorname{tg}\theta - d)\left(1 + \frac{1}{(z \operatorname{tg}\theta - d)^2 + z^2}\right) + \frac{d^2 + 1}{d} \\ \psi = z\left(1 - \frac{1}{(z \operatorname{tg}\theta - d)^2 + z^2}\right) \end{gathered} \tag{6.4}$$

By eliminating z from here one can find $\varphi_0(\psi) = \xi_0(\eta)$.

The longitudinal and transverse velocity components and the friction at the plate are found from Eqs. (5.5)-(5.9), in which one must set r equal to unity.

Let us examine the problem at the line of spreading flow $z = 0$. Here

$$\begin{gathered} \eta = 0, \quad \frac{\partial G}{\partial \lambda} = 0, \quad \xi = x\left(1 + \frac{1}{x^2}\right) + \frac{d^2 + 1}{d} \\ \xi_0 = 0, \quad U_e = 1 - 1/x^2 \end{gathered} \tag{6.5}$$

For the dimensionless friction we obtain

$$\left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=0} = \left(0.4146 \frac{\partial U_e}{\partial \xi} \frac{8}{U_e^{4/\pi+1}} \int_0^{\xi} U_e^{4/\pi} d\xi + 0.9368\right)\left(\frac{8}{\xi U_e^{4/\pi}} \int_0^{\xi} U_e^{4/\pi} d\xi\right)^{-1/2} \tag{6.6}$$

The pressure in the boundary layer along the line of spreading flow will increase as the cylinder is approached. Therefore the flow in the boundary layer is retarded and "separation" of the boundary layer develops. Let us find the point of separation on the line of spreading flow, i.e., the point at which the friction is reduced to zero. Since

$$dx = \frac{d\xi}{U_e}, \quad \frac{\partial U_e}{\partial \xi} = \frac{\partial U_e}{\partial x} \frac{1}{U_e}$$

the equation for the coordinate x of the point of separation takes the form

$$\frac{x^{8/\pi+1}}{(x^2 - 1)^{4/\pi+2}} \int_{-d}^{x} \left(1 - \frac{1}{x^2}\right)^{4/\pi+1} dx + 0.1412 = 0 \tag{6.7}$$

The location of the point of separation on the line of spreading flow depends only on d and does not depend on the slip angle θ.

The dependence of the location of the point of separation on the line of spreading flow on the distance of the cylinder axis from the leading edge of the plate (d), expressed through Eq. (6.7), is presented in Fig. 2. Here the experimental data of [7] and the data of the exact numerical calculations of [1] are given (the dash-dot curve is the present calculation, 2 is the experimental data, and 1 is the result of the finite-difference calculations). It is seen that both the experimental points and the exact solution fall nicely on the calculated curve. The dependence of the point of separation on the line of spreading flow on d in the first approximation for the locally self-similar solution is shown in the same figure. The locally self-similar solution is the approximate solution of the boundary layer equations when we neglect the derivatives of the unknown functions with respect to ξ and to η compared with the derivatives with respect to ζ, and the coordinates ξ and η enter into the equations only as parameters (the dashed curve in Fig. 2 gives the results of the locally self-similar approximation).

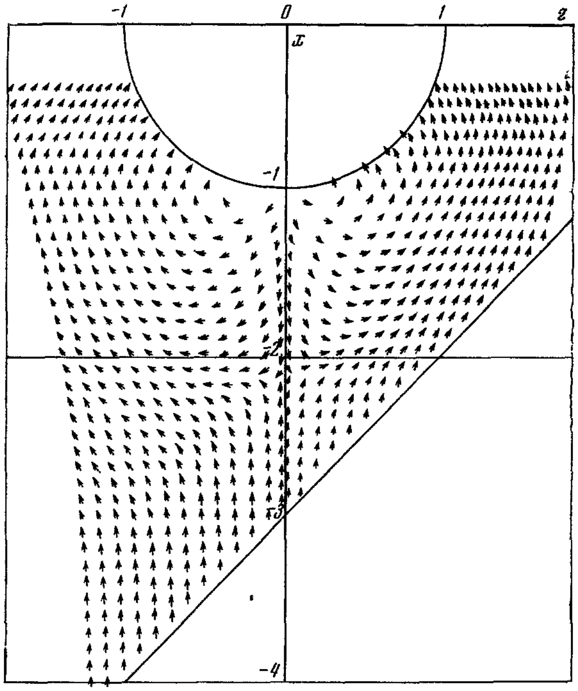

Fig. 5

The friction at the surface obtained from Eqs. (5.9) was compared with the results of exact numerical calculations and with the locally self-similar solution. The comparison was made for a slip angle to zero and d=4. It is seen in Fig. 3 that the first approximation agrees well with the exact solution.

A comparison of the dimensionless values of the friction components τ_x (curves 1) and τ_z (curves 2) obtained in the given approximation with the results of the finite-difference calculations and of the locally self-similar approximation are presented in Fig. 3 for x=−3 as a function of z.

The dimensionless friction component τ_x on the line of spreading flow z=0 as a function of d−x is presented in Fig. 3 (curve 3) (the dash-dot curve is the present calculation, the solid curve is the results of the finite-difference calculations, and the dashed curve is the locally self-similar approximation).

The behavior of the "limiting" streamlines at the body (d=3, θ=0) is shown in Fig. 4. The flow pattern in the boundary layer is completely different from the pattern of the external ideal flow. The "line of separation" is seen, which in this case is an envelope of the limiting streamlines.

The behavior of the limiting streamlines at the body in the presence of a slip angle (d=3, θ=45°) is shown in Fig. 5. The point of separation on the line of spreading flow does not depend on the slip angle, but the pattern of behavior of the streamlines is unsymmetrical in the two half-planes ($z > 0$, $z < 0$). The vortex which has formed can be noted beyond the line of separation. The proposed method makes it possible to find in the first approximation the pattern of behavior of the limiting streamlines everywhere on the body, beyond the line of separation in particular. The results obtained beyond the line of separation must be treated with some caution, since the boundary layer theory is approximate and the assumption that the secondary flow near the line of separation is small is not satisfied. Therefore these results are presented only partially, although they are of interest, since in our view they make it possible to explain the nature of the flow and to give a mathematically strict substantiation of the formulation of the problem in the entire region.

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